

Hawking radiation, W_∞ algebra and trace anomalies

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ABSTRACT: We apply the “trace anomaly method” to the calculation of moments of the Hawking radiation of a Schwarzschild black hole. We show that they can be explained as the fluxes of chiral currents forming a W_∞ algebra. Then we construct the covariant version of these currents and verify that up to order 6 they are not affected by any trace anomaly. Using cohomological methods we show that actually, for the fourth order current, no trace anomalies can exist. The results reported here are strictly valid in two dimensions.

KEYWORDS: Anomalies in Field and String Theories, Conformal and W Symmetry, Black Holes, 2D Gravity.

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1. Introduction

In the last few years there has been an increasing activity in calculating the Hawking radiation [1, 2] by means of anomalies. This renewed attention to the relation between anomalies and Hawking radiation was pioneered by the paper [3], which was followed by several other contributions [4–41]. In [3] the method used was based on the diffeomorphism anomaly in a two-dimensional effective field theory near the horizon of a radially symmetric static black hole. The argument is that, since just outside the horizon only outgoing modes may exist, the physics near the horizon can be described by an effective two-dimensional chiral field theory (of infinite many fields) in which ingoing modes have been integrated out. This implies an effective breakdown of the diff invariance. The ensuing anomaly equation can be utilized to compute the outgoing flux of radiation.

A different method, based on trace anomaly, had been suggested long ago by Christensen and Fulling, [42]. This method provides a full solution only in two dimensions, the reason being that its utilization involves the region away from the horizon, where a two-dimensional formalism does not provide a good description. The method has been reposed in different forms in [43, 44] and, in particular, [7] and [9]. In this paper we would like to discuss a few aspects of the trace anomaly method and its implications. In [7] the authors made the remarkable observation that the full spectrum of the Planck distribution of a thermal Hawking radiation of a Schwarzschild black hole can be described by postulating the existence, in the two-dimensional effective field theory near the horizon, of higher spin currents and applying a generalization of the trace anomaly method. These authors in subsequent papers fully developed this method for fermionic currents. In this

paper we do the same for bosonic higher spin currents. This allows us to clarify, first of all, that the higher spin currents necessary to reproduce the thermal Hawking radiation form a W_∞ algebra. We then covariantize the higher spin currents, according to the method proposed in [9], but, differently the latter reference, we do not find any trace anomaly in the higher spin currents. This prompts us to analyze the nature of these anomalies. Using consistency methods we find that the trace anomalies of ref. [9] are cohomologically trivial. This means that they are an artifact of the regularization employed.

2. W_∞ algebra and Hawking radiation

Let us review the argument that allows us to evaluate the outgoing radiation from a Schwarzschild black hole starting from the trace anomaly of the energy-momentum tensor (we closely follow [7]). Here we assume the point of view, advocated by several authors [43, 44] and in particular in [3], that near-horizon physics is described by a two-dimensional conformal field theory (see also [46, 45]). Due to the Einstein equation, the trace of the matter energy momentum tensor vanishes on shell. However it is generally the case that the latter is nonvanishing at one loop, due to an anomaly: $T_\alpha^\alpha = \frac{c}{24\pi}R$ where R is the background Ricci scalar. c is the central charge of the matter system. This is no accident, in fact it is well-known that the above trace anomaly is related to the cocycle that pops up in the conformal transformation of the (holomorphic or anti-holomorphic part of the) energy momentum tensor. If the matter system is chiral, this cocycle also determines the diffeomorphism anomaly (which we do not consider in this paper).

In light-cone coordinates $u = t - r_*$, $v = t + r_*$, let us denote by $T_{uu}(u, v)$ and $T_{vv}(u, v)$ the classically non vanishing components of the energy-momentum tensor. Given a background metric $g_{\alpha\beta} = e^\varphi \eta_{\alpha\beta}$, the trace anomaly equation (together with the conservation equation) can be solved. It yields

$$T_{uu}(u, v) = \frac{c}{24\pi} \left(\partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \right) + T_{uu}^{(hol)}(u) \tag{2.1}$$

where $T_{uu}^{(hol)}$ is holomorphic, while T_{uu} is conformally covariant. I.e., under a conformal transformation $u \rightarrow \tilde{u} = f(u)$ ($v \rightarrow \tilde{v} = g(u)$) one has

$$T_{uu}(u, v) = \left(\frac{df}{du} \right)^2 \tilde{T}_{\tilde{u}\tilde{u}}(\tilde{u}, \tilde{v}) \tag{2.2}$$

Since, under a conformal transformation, $\tilde{\varphi}(\tilde{u}, \tilde{v}) = \varphi(u, v) - \ln \left(\frac{df}{du} \frac{dg}{dv} \right)$, it follows that

$$T_{\tilde{u}\tilde{u}}^{(hol)}(\tilde{u}) = \left(\frac{df}{du} \right)^{-2} \left(T_{uu}^{(hol)}(u) + \frac{c}{24\pi} \{\tilde{u}, u\} \right) \tag{2.3}$$

Regular coordinates near the horizon are the Kruskal ones, (U, V) , defined by $U = -e^{-\kappa u}$ and $V = e^{\kappa v}$. Under this transformation we have

$$T_{UU}^{(hol)}(U) = \left(\frac{1}{\kappa U} \right)^{-2} \left(T_{uu}^{(hol)}(u) + \frac{c}{24\pi} \{U, u\} \right) \tag{2.4}$$

Now we require the outgoing energy flux to be regular at the horizon $U = 0$ in the Kruskal coordinate. Therefore at that point $T_{uu}^{(hol)}(u)$ is given by $\frac{c\kappa^2}{48\pi}$. Since the background is static, $T_{uu}^{(hol)}(u)$ is constant in t and therefore also in r . Therefore $\frac{c\kappa^2}{48\pi}$ is its value also at $r = \infty$. On the other hand we can assume that at $r = \infty$ there be no incoming flux and that the background be trivial (so that the vev of $T_{uu}^{(hol)}(u)$ and $T_{uu}(u, v)$ asymptotically coincide). Therefore the asymptotic flux is

$$\langle T_t^r \rangle = \langle T_{uu} \rangle - \langle T_{vv} \rangle = \frac{c\kappa^2}{48\pi} \tag{2.5}$$

Now let the thermal bosonic spectrum of the black hole, due to emission of a scalar complex boson ($c = 2$), be given by the Planck distribution

$$N(\omega) = \frac{2}{e^{\beta\omega} - 1} \tag{2.6}$$

where $1/\beta$ is the Hawking temperature and $\omega = |k|$. In two dimensions the flux moments are defined by

$$F_n = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dk \frac{\omega k^{n-2}}{e^{\beta\omega} - 1}$$

They vanish for n odd, while for n even they are given by

$$F_{2n} = \frac{1}{4\pi} \int_0^\infty d\omega \omega^{2n-1} N(\omega) = \frac{2(-1)^{n+1}}{8\pi n} B_{2n} \kappa^{2n} \tag{2.7}$$

where B_n are the Bernoulli numbers ($B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, \dots$). Therefore the outgoing flux (2.5) is seen to correspond to F_2 . The question posed by the authors of [7] was how to explain all the other moments. They suggested that this can be done in terms of higher tensorial currents. In other words the Hawking radiation flows to infinity carried by higher tensor generalizations of the energy-momentum tensor, which are coupled to suitable background fields that asymptotically vanish and do not back react.

The authors of [7, 9] used mostly higher spin currents bilinear in a fermionic field. They also suggested an analogous construction with other kinds of fields, and briefly discussed the case of a scalar bosonic field. In the following we would like to carry out the construction of higher spin currents in terms of a single complex bosonic field ($c = 2$). More explicitly, we will make use of the W_∞ algebra constructed by Bakas and Kiritsis long ago, [47]. To this end we go to the Euclidean and replace u, v with the complex coordinates z, \bar{z} .

2.1 The W_∞ algebra

Following [47] (see also [48–50]) we start with free complex boson having the following two point functions

$$\begin{aligned} \langle \phi(z_1) \bar{\phi}(z_2) \rangle &= -\log(z_1 - z_2) \\ \langle \phi(z_1) \phi(z_2) \rangle &= 0 \\ \langle \bar{\phi}(z_1) \bar{\phi}(z_2) \rangle &= 0 \end{aligned} \tag{2.8}$$

The currents are defined by

$$j_{z\dots z}^{(s)}(z) = B(s) \sum_{k=1}^{s-1} (-1)^k A_k^s : \partial_z^k \phi(z) \partial_z^{s-k} \bar{\phi}(z) : \quad (2.9)$$

where

$$B(s) = q^{s-2} \frac{2^{s-3} s!}{(2s-3)!!} \quad (2.10)$$

and

$$A_k^s = \frac{1}{s-1} \binom{s-1}{k} \binom{s-1}{s-k} \quad (2.11)$$

They satisfy a W_∞ algebra. The first several currents are

$$\begin{aligned} j_{zz}^{(2)} &= - : \partial_z \phi \partial_z \bar{\phi} : & (2.12) \\ j_{zzz}^{(3)} &= -2q (: \partial_z \phi \partial_z^2 \bar{\phi} : - : \partial_z^2 \phi \partial_z \bar{\phi} :) \\ j_{zzzz}^{(4)} &= -\frac{16q^2}{5} (: \partial_z \phi \partial_z^3 \bar{\phi} : - 3 : \partial_z^2 \phi \partial_z^2 \bar{\phi} : + : \partial_z^3 \phi \partial_z \bar{\phi} :) \\ j_{zzzzz}^{(5)} &= -\frac{32q^3}{7} (: \partial_z \phi \partial_z^4 \bar{\phi} : - 6 : \partial_z^2 \phi \partial_z^3 \bar{\phi} : + 6 : \partial_z^3 \phi \partial_z^2 \bar{\phi} : - : \partial_z^4 \phi \partial_z \bar{\phi} :) \\ j_{zzzzzz}^{(6)} &= -\frac{128q^4}{21} (: \partial_z \phi \partial_z^5 \bar{\phi} : - 10 : \partial_z^2 \phi \partial_z^4 \bar{\phi} : + 20 : \partial_z^3 \phi \partial_z^3 \bar{\phi} : - 10 : \partial_z^4 \phi \partial_z^2 \bar{\phi} : + : \partial_z^5 \phi \partial_z \bar{\phi} :) \end{aligned}$$

Normal ordering is defined as

$$: \partial^n \phi \partial^m \bar{\phi} : = \lim_{z_2 \rightarrow z_1} \{ \partial_{z_1}^n \phi(z_1) \partial_{z_2}^m \bar{\phi}(z_2) - \partial_{z_1}^n \partial_{z_2}^m \langle \phi(z_1) \bar{\phi}(z_2) \rangle \} \quad (2.13)$$

As usual in the framework of conformal field theory, the operator product in the r.h.s. is understood to be radial ordered.

The current $j_{zz}^{(2)}(z) = - : \partial_z \phi(z) \partial_z \bar{\phi}(z) :$ is proportional to the (normalized) holomorphic energy-momentum tensor of the model and, upon change of coordinates $z \rightarrow w(z)$, transforms as

$$: \partial_z \phi \partial_z \bar{\phi} : = (w')^2 : \partial_w \phi \partial_w \bar{\phi} : - \frac{1}{6} \{w, z\} \quad (2.14)$$

where $\{w, z\}$ — the Schwarzian derivative — is

$$\{w, z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2} \left(\frac{w''(z)}{w'(z)} \right)^2 \quad (2.15)$$

The non covariant contribution comes from the second term in (2.13). We have (see e.g. [51])

$$\begin{aligned} : \partial_{z_1} \phi(z_1) \partial_{z_2} \bar{\phi}(z_2) : &= \partial_{z_1} \phi(z_1) \partial_{z_2} \bar{\phi}(z_2) - \partial_{z_1} \partial_{z_2} \langle \phi(z_1) \bar{\phi}(z_2) \rangle & (2.16) \\ &= w'(z_1) w'(z_2) \partial_{w_1} \phi(w_1) \partial_{w_2} \bar{\phi}(w_2) - \partial_{z_1} \partial_{z_2} \langle \phi(z_1) \bar{\phi}(z_2) \rangle \\ &= w'(z_1) w'(z_2) : \partial_{w_1} \phi(w_1) \partial_{w_2} \bar{\phi}(w_2) : - G(z_1, z_2) \end{aligned}$$

where $\partial_{z_1}\phi(z_1)\partial_{z_2}\bar{\phi}(z_2)$ stands for the radial ordered product of the two operators, and

$$\begin{aligned}
 G(z_1, z_2) &= -w'(z_1)w'(z_2)\partial_{w_1}\partial_{w_2}\langle\phi(w_1)\bar{\phi}(w_2)\rangle + \partial_{z_1}\partial_{z_2}\langle\phi(z_1)\bar{\phi}(z_2)\rangle \\
 &= -\partial_{z_1}\partial_{z_2}(\langle\phi(w(z_1))\bar{\phi}(w(z_2))\rangle - \langle\phi(z_1)\bar{\phi}(z_2)\rangle) \\
 &= \frac{w'(z_1)w'(z_2)}{(w(z_1) - w(z_2))^2} - \frac{1}{(z_1 - z_2)^2}
 \end{aligned} \tag{2.17}$$

In the limit $z_2 \rightarrow z_1$ (2.17) becomes $\frac{1}{6}\{w, z_1\}$.

We are interested in the transformation properties of currents $j^{(s)}(u)$ when $w(z)$ is

$$w(z) = -e^{-\kappa z} \tag{2.18}$$

Analogously to (2.16), we have

$$j_{z_1 \dots z_s}^{(s)}(z_1) = \left(B(s) \sum_{k=1}^{s-1} (-1)^k A_k^s : \partial_{z_1}^k \phi(w(z_1)) \partial_{z_2}^{s-k} \bar{\phi}(w(z_2)) : \right) + \langle X \rangle_s \tag{2.19}$$

where

$$\begin{aligned}
 \langle X_s \rangle &= B(s) \sum_{k=1}^{s-1} (-1)^k A_k^s \lim_{z_2 \rightarrow z_1} \left\{ \langle \partial_{z_1}^k \phi(w(z_1)) \partial_{z_2}^{s-k} \bar{\phi}(w(z_2)) \rangle - \langle \partial_{z_1}^k \phi(z_1) \partial_{z_2}^{s-k} \bar{\phi}(z_2) \rangle \right\} \\
 &= \lim_{z_2 \rightarrow z_1} B(s) \sum_{k=1}^{s-1} (-1)^k A_k^s \partial_{z_1}^k \partial_{z_2}^{s-k} \left\{ \langle \phi(w(z_1)) \bar{\phi}(w(z_2)) \rangle - \langle \phi(z_1) \bar{\phi}(z_2) \rangle \right\} \\
 &= \lim_{z_2 \rightarrow z_1} B(s) \sum_{k=0}^{s-2} (-1)^{k+1} A_{k+1}^s \partial_{z_1}^k \partial_{z_2}^{s-k-2} \partial_{z_1} \partial_{z_2} \left\{ \langle \phi(w(z_1)) \bar{\phi}(w(z_2)) \rangle - \langle \phi(z_1) \bar{\phi}(z_2) \rangle \right\} \\
 &= B(s) \sum_{k=0}^{s-2} (-1)^k A_{k+1}^s \lim_{z_2 \rightarrow z_1} \partial_{z_1}^k \partial_{z_2}^{s-k-2} G(z_1, z_2) \\
 &= B(s) \sum_{k=0}^{s-2} (-1)^k A_{k+1}^s G_{k, s-k-2}
 \end{aligned} \tag{2.20}$$

and $G_{m,n}$ are coefficients in the series

$$G(z+a, z+b) = \sum_{m,n=0}^{\infty} \frac{a^m b^n}{m!n!} G_{m,n} \tag{2.21}$$

We now evaluate coefficients for the transformation (2.18). Putting (2.18) in (2.17) we obtain

$$G(z_1, z_2) = G(z_1 - z_2) = -\frac{1}{(z_1 - z_2)^2} + \frac{\kappa^2}{4} \frac{1}{\sinh^2 \frac{\kappa(z_1 - z_2)}{2}} \tag{2.22}$$

This gives¹

$$G_{m,n} = (-)^{n+1} \kappa^{m+n+2} \frac{B_{m+n+2}}{m+n+2} \quad (2.23)$$

So, we obtain

$$\langle X_s \rangle = (-)^{s-1} (4q)^{s-2} \kappa^s \frac{B_s}{s} \quad (2.24)$$

We have used

$$\sum_{k=0}^{s-2} A_{k+1}^s = \frac{(2s-2)!}{(s-1)!s!} \quad (2.25)$$

(2.24) is a higher order Schwarzian derivative evaluated at $w(z) = -e^{-\kappa z}$. It plays a role analogous to the r.h.s. of (2.5). In the next section we will compare it with the radiation moments in the r.h.s. of (2.7).

2.2 Higher moments of the black hole radiation

Let us now return to the light-cone notation. We identify $j_{uu}^{(2)}(u)$ up to a constant² with the holomorphic energy momentum tensor

$$j_{uu}^{(2)}(u) = -2\pi T_{uu}^{(hol)} \quad (2.26)$$

Similarly we identify $j_{u\dots u}^{(s)}$, with s lower indices, with an s -th order holomorphic tensor. They can be naturally thought of as the only non-vanishing components of a two-dimensional completely symmetric current. In analogy with the energy-momentum tensor, we expect that there exist a conformally covariant version $J_{u\dots u}^{(s)}$ of $j_{u\dots u}^{(s)}$. The latter must be the intrinsic component of a two-dimensional completely symmetric traceless current $J_{\mu_1\dots\mu_s}^{(s)}$, whose only other classically non-vanishing component is $J_{v\dots v}^{(s)}$.

Now let us apply to these currents an argument similar to the one in section 2 for the energy-momentum tensor, using the previous results from the W_∞ algebra. Introducing the Kruskal coordinate $U = -e^{-\kappa u}$ and requiring regularity at the horizon we find that, at the horizon, the value of $j_{u\dots u}^{(s)}$ is given by $\langle X_s \rangle$ in eq. (2.24). Next $j_{u\dots u}^{(s)}(u)$ is constant in t and r (the same is of course true for $j_{v\dots v}^{(s)}$). Therefore, if we identify $j_{u\dots u}^{(s)}(u)$ with $j_{z\dots z}^{(s)}(z)$ via Wick rotation, $\langle X_s \rangle$ corresponds to its value at $r = \infty$. Since $j_{u\dots u}^{(s)}(u)$ and $J_{u\dots u}^{(s)}(u)$ asymptotically coincide, the asymptotic flux of this current is

$$-\frac{1}{2\pi} \langle J^{(s)r}{}_{t\dots t} \rangle = -\frac{1}{2\pi} \langle J_{u\dots u}^{(s)} \rangle + \frac{1}{2\pi} \langle J_{v\dots v}^{(s)} \rangle = -\frac{1}{2\pi} \langle X_s \rangle = \frac{i^{s-2}}{2\pi s} \kappa^s B_s \quad (2.27)$$

¹Note that

$$\begin{aligned} -\frac{1}{x^2} + \frac{\kappa^2}{4} \frac{1}{\sinh^2 \frac{\kappa x}{2}} &= \frac{d}{dx} \left(\frac{1}{x} \left(1 - \frac{\kappa x}{e^{\kappa x} - 1} \right) \right) = \frac{d}{dx} \left(\frac{1}{x} \left(1 - \sum_{n=0}^{\infty} B_n \frac{(\kappa x)^n}{n!} \right) \right) \\ &= -\kappa^2 \sum_{n=2}^{\infty} (n-1) B_n \frac{(\kappa x)^{n-2}}{n!} = -\kappa^2 \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{(\kappa x)^n}{n!} \end{aligned}$$

²We relate $j_{uu}^{(2)}$ with the energy momentum tensor via the factor of 2π and the minus sign. This is because in the Euclidean we want to conform to the conventions and results of [47], where properly normalized currents satisfy a W_∞ algebra. This holds for higher order currents too: for physical applications their W_∞ representatives must all be divided by -2π .

provided we set the deformation parameter q to the value $-\frac{i}{4}$ (for the global -2π factor, see the previous footnote).

The r.h.s. vanishes for odd s (except $s = 1$ which is not excited in our case) and coincides with the thermal flux moments (2.7) for even s .

It remains for us to show that the covariant conserved currents $J_{\mu_1 \dots \mu_s}^{(s)}$ can be defined.

3. Higher spin covariant currents

To start with, it is natural to suppose that the covariant currents appear in an effective action S where they are sourced by asymptotically trivial background fields $B_{\mu_1 \dots \mu_s}^{(s)}$ (in [52] they were called ‘cometric functions’), i.e.

$$J_{\mu_1 \dots \mu_s}^{(s)} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta B_{\mu_1 \dots \mu_s}^{(s)}} S \tag{3.1}$$

In particular $B_{\mu\nu}^{(2)} = g_{\mu\nu}/2$. We assume that all $J_{\mu_1 \dots \mu_s}^{(s)}$ are maximally symmetric and classically traceless.

In order to find a covariant expression we first recall that the previous W_∞ algebra is formulated in terms of a (complex, Euclidean) chiral bosonic field. The action of a chiral (Minkowski) scalar in 2D coupled to background gravity can be found in [54]. When the background gravity is of the type considered in this paper, i.e. $g_{\alpha\beta} = e^\varphi \eta_{\alpha\beta}$, the action boils down to that of a free chiral boson, [53]. In other words, the equation of motion of a chiral boson coupled to background conformal gravity is

$$\partial_v \phi = 0 \tag{3.2}$$

This simplifies the covariantization process.

To proceed with the covariantization program we then reduce the problem to a one-dimensional one. We consider only the u dependence and keep v fixed. In one dimension a curved coordinate u is easily related to the corresponding normal coordinate x via the relation $\partial_x = e^{-\varphi(u)} \partial_u$. We view u as $u(x)$, assume that all $j_{u \dots u}^{(s)}$ and their W_∞ relations refer in fact to the flat x coordinate (i.e. x corresponds to the Euclidean coordinate z used in the previous section) and by the above equivalence we extract the components in the new coordinate system. For instance for a scalar field ϕ :

$$\partial_x^n \phi = e^{-n\varphi(u)} \nabla_u^n \phi, \quad \text{i.e.} \quad \partial_x^n \phi (dx)^n = \nabla_u^n \phi (du)^n$$

We recall that the W_∞ currents are constructed out of bilinears in ϕ and $\bar{\phi}$:

$$j_{u \dots u}^{(n,m)} = : \partial_u^n \phi \partial_u^m \bar{\phi} : \tag{3.3}$$

We split the factors and evaluate one factor in $u_1 = u(x + \epsilon/2)$ and the other in $u_2 = u(x - \epsilon/2)$. We expand in ϵ and take the limit for $\epsilon \rightarrow 0$. Afterward we restore the tensorial character of the product by multiplying it by a suitable $e^{n\varphi(u)}$ factor. We use in particular the Taylor expansion, see [9],

$$u(x + \epsilon) = u(x) + \epsilon e^{-\varphi} - \frac{\epsilon^2}{2} e^{-2\varphi} \partial_u \varphi + \dots$$

According to the recipe just explained, the covariant counterpart of $j_{u\dots u}^{(s)}$ should be constructed using currents

$$J_{u\dots u}^{(n,m)} = e^{(n+m)\varphi(u)} \lim_{\epsilon \rightarrow 0} \left\{ e^{-n\varphi(u_1) - m\varphi(u_2)} \nabla_{u_1}^n \phi \nabla_{u_2}^m \bar{\phi} - \frac{c_{n,m} \hbar}{\epsilon^{n+m}} \right\} \quad (3.4)$$

where $c_{n,m} = (-)^m (n+m-1)!$ are numerical constants determined in such a way that all singularities are canceled in the final expression for $J_{u\dots u}^{(n,m)}$. Therefore (3.4) defines the normal ordered current

$$J_{u\dots u}^{(n,m)} = : \nabla_u^n \phi \nabla_u^m \bar{\phi} : \quad (3.5)$$

We use

$$\nabla_u \nabla_u^n f(u, v) = \partial_u \nabla_u^n f(u, v) - n \Gamma \nabla_u^n f(u, v)$$

for a scalar field $f(u, v)$, where

$$\Gamma = \partial_u \varphi$$

and

$$: \phi(u_1) \bar{\phi}(u_2) : = \phi(u_1) \bar{\phi}(u_2) + \hbar \log(u_1 - u_2) \quad (3.6)$$

After some algebra we obtain

$$\begin{aligned} J_{uu}^{(1,1)} &= \frac{\hbar}{6} T + j_{uu}^{(1,1)} \\ J_{uuu}^{(1,2)} &= \frac{\hbar}{12} (\partial_u T) - \Gamma J_{uu}^{(1,1)} + j_{uuu}^{(1,2)} \\ J_{uuu}^{(2,1)} &= \frac{\hbar}{12} (\partial_u T) - \Gamma J_{uu}^{(1,1)} + j_{uuu}^{(2,1)} \\ J_{uuuu}^{(1,3)} &= \frac{\hbar}{20} (\partial_u^2 T) + \frac{\hbar}{30} T^2 - J_{uu}^{(1,1)} T - \frac{3}{2} \Gamma^2 J_{uu}^{(1,1)} - 3 \Gamma J_{uuu}^{(1,2)} + j_{uuuu}^{(1,3)} \\ J_{uuuu}^{(2,2)} &= \frac{\hbar}{30} (\partial_u^2 T) - \frac{\hbar}{30} T^2 - \Gamma^2 J_{uu}^{(1,1)} - \Gamma J_{uuu}^{(1,2)} - \Gamma J_{uuu}^{(2,1)} + j_{uuuu}^{(2,2)} \\ J_{uuuu}^{(3,1)} &= \frac{\hbar}{20} (\partial_u^2 T) + \frac{\hbar}{30} T^2 - J_{uu}^{(1,1)} T - \frac{3}{2} \Gamma^2 J_{uu}^{(1,1)} - 3 \Gamma J_{uuu}^{(2,1)} + j_{uuuu}^{(3,1)} \end{aligned} \quad (3.7)$$

where

$$T = \partial_u^2 \varphi - \frac{1}{2} (\partial_u \varphi)^2 \quad (3.8)$$

In appendix one can find analogous expressions for order 5 and 6 currents.

Using eq. (2.12), and similarly, $J_{uu}^{(2)} = -J_{uu}^{(1,1)}$, $J_{uuu}^{(3)} = -2q (J_{uuu}^{(1,2)} - J_{uuu}^{(2,1)})$, $J_{uuuu}^{(4)} = -\frac{16q^2}{5} (J_{uuuu}^{(1,3)} - 3J_{uuuu}^{(2,2)} + J_{uuuu}^{(3,1)})$, etc., we obtain

$$\begin{aligned} J_{uu}^{(2)} &= j_{uu}^{(2)} - \frac{\hbar}{6} T \\ J_{uuu}^{(3)} &= j_{uuu}^{(3)} \\ J_{uuuu}^{(4)} &= j_{uuuu}^{(4)} - \frac{8\hbar}{15} q^2 T^2 - \frac{32}{5} q^2 T J_{uu}^{(2)} \\ J_{uuuuu}^{(5)} &= j_{uuuuu}^{(5)} - \frac{160}{7} q^2 T J_{uuu}^{(3)} \end{aligned} \quad (3.9)$$

For $s = 6$ we have

$$\begin{aligned}
 J_{uuuuuu}^{(6)} = & \left(-\frac{512\hbar}{63}T^3 + \frac{160\hbar}{63}(\partial_u T)^2 - \frac{128\hbar}{63}T\partial_u^2 T - \frac{512}{3}T^2 J_{uu}^{(2)} - \frac{256}{21}T\nabla_u^2 J_{uu}^{(2)} \right. \\
 & - \frac{256}{21}(\partial_u^2 T) J_{uu}^{(2)} + \frac{640}{21}(\partial_u T)\nabla_u J_{uu}^{(2)} - \frac{1280}{21}\Gamma T\nabla_u J_{uu}^{(2)} - \frac{1280}{21}\Gamma^2 T J_{uu}^{(2)} \\
 & \left. + \frac{1280}{21}\Gamma(\partial_u T) J_{uu}^{(2)} \right) q^4 - \frac{160}{3}q^2 T J_{uuuu}^{(4)} + j_{uuuuuu}^{(6)} \quad (3.10)
 \end{aligned}$$

It is important to verify that our previous definitions are consistent. Using the transformation law for $j_{uu}^{(2)}$ (i.e. (2.14))

$$j_{uu}^{(2)}(u) = (w'(u))^2 \tilde{j}_{ww}^{(2)}(w(u)) + \frac{\hbar}{6} \{w, u\} \quad (3.11)$$

and its generalization for $j_{uuuu}^{(4)}$ which can be read out of (2.19)

$$j_{uuuu}^{(4)}(u) = (w'(u))^4 \tilde{j}_{wwww}^{(4)}(w(u)) + q^2 (w')^2 \frac{32}{5} \tilde{j}_{ww}^{(2)}(w(u)) \{w, u\} + \hbar q^2 \frac{8}{15} \{w, u\}^2 \quad (3.12)$$

and using

$$\varphi(u, v) = \tilde{\varphi}(w(u), v) + \log(w'(u)) \quad (3.13)$$

it can be checked that $J_{uu}^{(2)}$ and $J_{uuuu}^{(4)}$ transform indeed as tensors

$$\begin{aligned}
 J_{uu}^{(2)}(u) &= (w'(u))^2 \tilde{J}_{ww}^{(2)}(w(u)) \\
 J_{uuuu}^{(4)}(u) &= (w'(u))^4 \tilde{J}_{wwww}^{(4)}(w(u))
 \end{aligned} \quad (3.14)$$

The next step consists in finding the covariant derivatives of the currents. The only v dependence comes from φ . We have

$$\begin{aligned}
 g^{uv} \nabla_v J_{uu}^{(1,1)} &= -\frac{\hbar}{12} (\nabla_u R) \\
 g^{uv} \nabla_v J_{uuu}^{(1,2)} &= -\frac{\hbar}{24} (\nabla_u^2 R) + \frac{1}{2} R J_{uu}^{(1,1)} \\
 g^{uv} \nabla_v J_{uuu}^{(2,1)} &= -\frac{\hbar}{24} (\nabla_u^2 R) + \frac{1}{2} R J_{uu}^{(1,1)} \\
 g^{uv} \nabla_v J_{uuuu}^{(1,3)} &= -\frac{\hbar}{40} (\nabla_u^3 R) + \frac{1}{2} (\nabla_u R) J_{uu}^{(1,1)} + \frac{3}{2} R J_{uuu}^{(1,2)} \\
 g^{uv} \nabla_v J_{uuuu}^{(2,2)} &= -\frac{\hbar}{60} (\nabla_u^3 R) + \frac{1}{2} R J_{uuu}^{(1,2)} + \frac{1}{2} R J_{uuu}^{(2,1)} \\
 g^{uv} \nabla_v J_{uuuu}^{(3,1)} &= -\frac{\hbar}{40} (\nabla_u^3 R) + \frac{1}{2} (\nabla_u R) J_{uu}^{(1,1)} + \frac{3}{2} R J_{uuu}^{(2,1)}
 \end{aligned} \quad (3.15)$$

and, using (A.1) and (A.2) in appendix,

$$\begin{aligned}
 g^{uv} \nabla_v J_{uuuuu}^{(1,4)} &= -\frac{\hbar}{60} (\nabla_u^4 R) + \frac{1}{2} (\nabla_u^2 R) J_{uu}^{(1,1)} + 2 (\nabla_u R) J_{uuu}^{(1,2)} + 3 R J_{uuuu}^{(1,3)} \\
 g^{uv} \nabla_v J_{uuuuu}^{(2,3)} &= -\frac{\hbar}{120} (\nabla_u^4 R) + \frac{1}{2} (\nabla_u R) J_{uuu}^{(2,1)} + \frac{1}{2} R J_{uuuu}^{(1,3)} + \frac{3}{2} R J_{uuuu}^{(2,2)} \\
 g^{uv} \nabla_v J_{uuuuu}^{(3,2)} &= -\frac{\hbar}{120} (\nabla_u^4 R) + \frac{1}{2} (\nabla_u R) J_{uuu}^{(1,2)} + \frac{1}{2} R J_{uuuu}^{(3,1)} + \frac{3}{2} R J_{uuuu}^{(2,2)}
 \end{aligned}$$

$$\begin{aligned}
 g^{uv}\nabla_v J_{uuuu}^{(4,1)} &= -\frac{\hbar}{60}(\nabla_u^4 R) + \frac{1}{2}(\nabla_u^2 R) J_{uu}^{(1,1)} + 2(\nabla_u R) J_{uuu}^{(2,1)} + 3R J_{uuuu}^{(3,1)} \\
 g^{uv}\nabla_v J_{uuuuuu}^{(1,5)} &= -\frac{\hbar}{84}(\nabla_u^5 R) + \frac{1}{2}(\nabla_u^3 R) J_{uu}^{(1,1)} + \frac{5}{2}(\nabla_u^2 R) J_{uuu}^{(1,2)} \\
 &\quad + 5(\nabla_u R) J_{uuuu}^{(1,3)} + 5R J_{uuuuu}^{(1,4)} \\
 g^{uv}\nabla_v J_{uuuuuu}^{(2,4)} &= -\frac{\hbar}{210}(\nabla_u^5 R) + \frac{1}{2}(\nabla_u^2 R) J_{uuu}^{(2,1)} + 2(\nabla_u R) J_{uuuu}^{(2,2)} \\
 &\quad + \frac{1}{2}R J_{uuuuu}^{(1,4)} + 3R J_{uuuuu}^{(2,3)} \\
 g^{uv}\nabla_v J_{uuuuuu}^{(3,3)} &= -\frac{\hbar}{280}(\nabla_u^5 R) + \frac{1}{2}(\nabla_u R) J_{uuuu}^{(1,3)} + \frac{1}{2}(\nabla_u R) J_{uuuu}^{(3,1)} \\
 &\quad + \frac{3}{2}R J_{uuuuu}^{(2,3)} + \frac{3}{2}R J_{uuuuu}^{(3,2)} \\
 g^{uv}\nabla_v J_{uuuuuu}^{(4,2)} &= -\frac{\hbar}{210}(\nabla_u^5 R) + \frac{1}{2}(\nabla_u^2 R) J_{uuu}^{(1,2)} + 2(\nabla_u R) J_{uuuu}^{(2,2)} \\
 &\quad + \frac{1}{2}R J_{uuuuu}^{(4,1)} + 3R J_{uuuuu}^{(3,2)} \\
 g^{uv}\nabla_v J_{uuuuuu}^{(5,1)} &= -\frac{\hbar}{84}(\nabla_u^5 R) + \frac{1}{2}(\nabla_u^3 R) J_{uu}^{(1,1)} + \frac{5}{2}(\nabla_u^2 R) J_{uuu}^{(2,1)} \\
 &\quad + 5(\nabla_u R) J_{uuuu}^{(3,1)} + 5R J_{uuuuu}^{(4,1)}
 \end{aligned}$$

For the currents $J_{u\dots u}^{(s)}$, which are the linear combinations of $J_{u\dots u}^{(n,m)}$ we obtain

$$g^{uv}\nabla_v J_{uu}^{(2)} = \frac{\hbar}{12}(\nabla_u R) \quad (3.16)$$

$$g^{uv}\nabla_v J_{uuu}^{(3)} = 0 \quad (3.17)$$

$$g^{uv}\nabla_v J_{uuuu}^{(4)} = \frac{16}{5}q^2(\nabla_u R) J_{uu}^{(2)} \quad (3.18)$$

$$g^{uv}\nabla_v J_{uuuuu}^{(5)} = \frac{80}{7}q^2(\nabla_u R) J_{uuu}^{(3)} \quad (3.19)$$

For $s = 6$:

$$\begin{aligned}
 g^{uv}\nabla_v J_{uuuuuu}^{(6)} &= \left(-\frac{320}{21}(\nabla_u^2 R) \nabla_u J_{uu}^{(2)} + \frac{128}{21}(\nabla_u R) \nabla_u^2 J_{uu}^{(2)} + \frac{128}{21}(\nabla_u^3 R) J_{uu}^{(2)} \right) q^4 \\
 &\quad + \frac{80}{3}(\nabla_u R) J_{uuuu}^{(4)} q^2
 \end{aligned} \quad (3.20)$$

Now, according to [9], after the right hand side is expressed in terms of covariant quantities, terms proportional to \hbar are identified as anomalies in the following way. One assumes that there is no anomaly in the conservation laws of covariant currents, i.e that the terms proportional to \hbar do not appear in $\nabla^\mu J_{\mu u\dots u}$. Since $\nabla^\mu J_{\mu u\dots u} = g^{uv}\nabla_v J_{uu\dots u} + g^{uv}\nabla_u J_{vu\dots u}$, one relates terms proportional to \hbar in the u derivative of the trace ($vu\dots u$ components) with the terms proportional to \hbar in the v derivative of $uu\dots u$ components of the currents.

For the covariant energy momentum tensor $J_{\mu\nu}^{(2)}$, the trace is $\text{Tr}(J^{(2)}) = 2g^{vu} J_{vu}^{(2)}$. Thus, (3.16) reproduces the well known trace anomaly $\text{Tr}(J^{(2)}) = -\frac{c\hbar}{12}R$, where in our case $c = 2$ (for the missing factor of -2π see the footnote in section 2.2).

We see that the terms that carry explicit factors of \hbar cancel out in eqs. (3.17)-(3.20). This implies the absence of \hbar in the trace, and consequently the absence of the trace anomaly.

4. Trace anomalies

In the previous section the covariant form of the current does not give rise to any trace anomaly. This is at variance with ref. [9], where the fourth order covariantized current exhibits a trace anomaly which is a superposition of three terms: $\nabla_\mu \nabla_\nu R$, $g_{\mu\nu} \square R$ and $g_{\mu\nu} R^2$. It is therefore important to clarify whether these are true anomalies or whether they are some kind of artifact of the regularization used to derive the results.

In the framework of the effective action introduced in the previous section (see (3.1)), the anomaly problem can be clarified using cohomological (or consistency) methods. Such methods were applied for the first time to the study of trace anomalies in [55, 57]. Subsequent applications can be found in [56, 58] and more recently in [59, 60]. The consistency conditions for trace anomalies are similar to the Wess-Zumino consistency conditions for chiral anomalies and are based on the simple remark that, if we perform two symmetry transformations in different order on the one-loop action, the result must obey the group theoretical rules of the transformations. In particular, since Weyl transformations are Abelian, making two Weyl transformations in opposite order must bring the same result. Although this explains the geometrical meaning of the consistency conditions, proceeding in this way is often very cumbersome. The problem becomes more manageable if we transform it into a cohomological one. This is simple: just promote the local transformation parameters to anticommuting fields (ghost). The transformations become nilpotent and define a coboundary operator.

In this section we will consider, for simplicity, the possible anomalies of the fourth order current $J_{\mu\nu\lambda\rho}^{(4)}$ which couples in the action to the background field $B_{\mu\nu\lambda\rho}^{(4)} \equiv B_{\mu\nu\lambda\rho}$, both being completely symmetric tensors. The relevant Weyl transformations are as follows. The gauge parameters are the usual Weyl parameter σ and new Weyl parameters $\tau_{\mu\nu}$ (symmetric in μ, ν). The variation δ_τ acts only on $B_{\mu\nu\lambda\rho}$ (see [52])

$$\delta_\tau B_{\mu\nu\lambda\rho} = g_{\mu\nu} \tau_{\lambda\rho} + g_{\mu\lambda} \tau_{\nu\rho} + g_{\mu\rho} \tau_{\nu\lambda} + g_{\nu\lambda} \tau_{\mu\rho} + g_{\nu\rho} \tau_{\mu\lambda} + g_{\lambda\rho} \tau_{\mu\nu} \quad (4.1)$$

while δ_σ acts on $g_{\mu\nu}$, $\tau_{\mu\nu}$ and $B_{\mu\nu\lambda\rho}$ in the following way

$$\begin{aligned} \delta_\sigma g_{\mu\nu} &= 2 \sigma g_{\mu\nu} \\ \delta_\sigma \tau_{\mu\nu} &= (x - 2) \sigma \tau_{\mu\nu} \\ \delta_\sigma B_{\mu\nu\lambda\rho} &= x \sigma B_{\mu\nu\lambda\rho} \end{aligned} \quad (4.2)$$

where x is a free numerical parameter. The transformation (4.3) of τ and B are required for consistency with (4.1). The actual value of x turns out to be immaterial.

Now we promote σ and τ to anticommuting fields:

$$\begin{aligned} \sigma^2 &= 0 \\ \tau_{\mu\nu} \tau_{\lambda\rho} + \tau_{\lambda\rho} \tau_{\mu\nu} &= 0 \\ \sigma \tau_{\mu\nu} + \tau_{\mu\nu} \sigma &= 0 \end{aligned}$$

It is easy to verify that

$$\delta_\sigma^2 = 0, \quad \delta_\tau^2 = 0, \quad \delta_\sigma \delta_\tau + \delta_\tau \delta_\sigma = 0$$

Therefore they define a double complex.

Integrated anomalies are defined by

$$\delta_\sigma \Gamma^{(1)} = \hbar \Delta_\sigma, \quad \delta_\tau \Gamma^{(1)} = \hbar \Delta_\tau, \quad (4.3)$$

where $\Gamma^{(1)}$ is the one-loop quantum action and $\Delta_\sigma, \Delta_\tau$ are local functional linear in σ and τ , respectively. The unintegrated anomalies, i.e. the traces T_μ^μ and $J^{(4)\mu}{}_{\mu\lambda\rho}$ are obtained by functionally differentiating with respect to σ and $\tau_{\lambda\rho}$, respectively.

By applying $\delta_\sigma, \delta_\tau$ to the eqs. (4.3), we see that candidates for anomalies Δ_σ and Δ_τ must satisfy the consistency conditions

$$\delta_\sigma \Delta_\sigma = 0 \quad (4.4)$$

$$\delta_\tau \Delta_\sigma + \delta_\sigma \Delta_\tau = 0 \quad (4.5)$$

$$\delta_\tau \Delta_\tau = 0 \quad (4.6)$$

i.e. they must be cocycles. We have to make sure that they are true anomalies, that is that they are nontrivial. In other words there must not exist local counterterm C in the action such that

$$\Delta_\sigma = \delta_\sigma \int d^2x C \quad (4.7)$$

$$\Delta_\tau = \delta_\tau \int d^2x C \quad (4.8)$$

If such a C existed we could redefine the quantum action by subtracting these counterterms and get rid of the (trivial) anomalies.

We start by expanding candidate anomalies as linear combinations of curvature invariants³

$$\Delta_\sigma = \int d^2x \sqrt{-g} \sum_{i=2}^{11} c_i I_i \quad (4.10)$$

$$\Delta_\tau = \int d^2x \sqrt{-g} \sum_{k=1}^3 b_k K_k \quad (4.11)$$

³The fact that we are in 2 spacetime dimensions reduces greatly the number of curvature invariants, such as those in (4.12). Useful relations valid in 2 dimensions are

$$\begin{aligned} R_{\mu\nu\lambda\rho} &= \frac{1}{2} R (g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda}) \\ R_{\mu\nu} &= \frac{1}{2} g_{\mu\nu} R \\ \delta_\sigma R &= -2 R \sigma - 2 \square \sigma \end{aligned} \quad (4.9)$$

where I_i are linear in $B^{\mu\nu\lambda\rho}$ and σ :

$$\begin{aligned}
 I_1 &= \sigma R & (4.12) \\
 I_2 &= B^{\mu\nu\lambda\rho} \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\rho \sigma \\
 I_3 &= B^{\mu\nu} R \nabla_\mu \nabla_\nu \sigma \\
 I_4 &= B^{\mu\nu} \nabla_\mu \nabla_\nu \square \sigma \\
 I_5 &= B^{\mu\nu} \nabla_\mu \nabla_\nu R \sigma \\
 I_6 &= B \square R \sigma \\
 I_7 &= B R^2 \sigma \\
 I_8 &= B^{\mu\nu} \nabla_\mu R \nabla_\nu \sigma \\
 I_9 &= B R \square \sigma \\
 I_{10} &= B g^{\mu\nu} \nabla_\mu R \nabla_\nu \sigma \\
 I_{11} &= B \square^2 \sigma
 \end{aligned}$$

($B^{\mu\nu} = B^{\mu\nu\lambda\rho} g_{\lambda\rho}$, $B = B^{\mu\nu} g_{\mu\nu}$). The term I_1 corresponds to the usual anomaly of the energy-momentum trace (which is consistent and nontrivial). Therefore in the sequel we disregard it and limit ourselves to the other terms which contain 4 derivatives. Similarly K_k are independent curvature invariants that are linear in $\tau_{\mu\nu}$ and contain 4 derivatives:

$$\begin{aligned}
 K_1 &= \nabla_\mu \nabla_\nu R \tau^{\mu\nu} & (4.13) \\
 K_2 &= R^2 \tau \\
 K_3 &= \square R \tau
 \end{aligned}$$

where $\tau = g^{\mu\nu} \tau_{\mu\nu}$.

Now we apply the consistency condition (4.4) to Δ_σ in the form (4.10). We obtain

$$\delta_\sigma \Delta_\sigma = \sum_{i=2}^{11} \sum_{j=1}^{12} c_i A_{ij} \int d^2x \sqrt{-g} J_j^{\sigma\sigma} = 0 \quad (4.14)$$

where the variations $\delta_\sigma I_i$ are expressed as linear combinations of terms $J_j^{\sigma\sigma}$

$$\begin{aligned}
 J_1^{\sigma\sigma} &= B^{\mu\nu} R \sigma \nabla_\mu \nabla_\nu \sigma & (4.15) \\
 J_2^{\sigma\sigma} &= B^{\mu\nu} \nabla_\mu R \sigma \nabla_\nu \sigma \\
 J_3^{\sigma\sigma} &= B R \sigma \square \sigma \\
 J_4^{\sigma\sigma} &= B g^{\mu\nu} \nabla_\mu R \sigma \nabla_\nu \sigma \\
 J_5^{\sigma\sigma} &= B^{\mu\nu\lambda\rho} \sigma \nabla_\mu \nabla_\nu \nabla_\lambda \nabla_\rho \sigma \\
 J_6^{\sigma\sigma} &= B^{\mu\nu\lambda\rho} \nabla_\mu \sigma \nabla_\nu \nabla_\lambda \nabla_\rho \sigma \\
 J_7^{\sigma\sigma} &= B^{\mu\nu} \sigma \nabla_\mu \nabla_\nu \square \sigma \\
 J_8^{\sigma\sigma} &= B^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \square \sigma \\
 J_9^{\sigma\sigma} &= B^{\mu\nu} \square \sigma \nabla_\mu \nabla_\nu \sigma \\
 J_{10}^{\sigma\sigma} &= B^{\mu\nu} g^{\lambda\rho} \nabla_\lambda \sigma \nabla_\mu \nabla_\nu \nabla_\rho \sigma
 \end{aligned}$$

$$\begin{aligned}
 J_{11}^{\sigma\sigma} &= B \sigma \square^2 \sigma \\
 J_{12}^{\sigma\sigma} &= B g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \square \sigma
 \end{aligned}$$

with coefficients given by

$$A_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & x-6 & -10 & 0 & 0 & 0 & 5 & 0 & 0 \\ x-6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x-6 & -6 & 2 & 0 & 0 & 1 \\ 2 & 6 & 0 & -1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x-6 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x-6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x-6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x-6 & -4 \end{pmatrix} \quad (4.16)$$

This gives a homogeneous system of equations for c_2, \dots, c_{11}

$$\sum_{i=2}^{11} c_i A_{ij} = 0, \quad j = 1, \dots, 12 \quad (4.17)$$

The solution can be expressed in terms of 3 free parameters which we take to be c_9, c_{10}, c_{11} . We have

$$\begin{aligned}
 c_2 &= 0 \\
 c_3 &= -2(c_{10} - 2c_{11}) \\
 c_4 &= -2(c_{10} - 2c_{11}) \\
 c_5 &= (c_{10} - 2c_{11})(x - 6) \\
 c_6 &= -\frac{1}{2}c_{11}(x - 6) \\
 c_7 &= \frac{1}{4}(x - 6)(c_{11} - c_9) \\
 c_8 &= -6(c_{10} - 2c_{11})
 \end{aligned} \quad (4.18)$$

Now we plug this solution (4.18) back into (4.10) and apply the consistency condition (4.5)

$$\begin{aligned}
 \delta_\tau \Delta_\sigma + \delta_\sigma \Delta_\tau &= \delta_\tau \left(\int d^2x \sqrt{-g} \sum_{i=2}^{12} c_i I_i \right) + \delta_\sigma \left(\int d^2x \sqrt{-g} \sum_{k=1}^3 b_k K_k \right) \\
 &= \int d^2x \sqrt{-g} \sum_{j=1}^9 \left(\sum_{i=9}^{11} c_i A_{ij}^{\tau\sigma} + \sum_{k=1}^3 b_k A_{kj}^{\sigma\tau} \right) J_j^{\tau\sigma} = 0
 \end{aligned} \quad (4.19)$$

Here the result of the variations is expressed as linear combinations of the curvature invariants denoted by $J_j^{\tau\sigma}$:

$$\begin{aligned}
 J_1^{\tau\sigma} &= \tau^{\mu\nu} \nabla_\mu \nabla_\nu \square \sigma \\
 J_2^{\tau\sigma} &= R \tau^{\mu\nu} \nabla_\mu \nabla_\nu \sigma
 \end{aligned} \quad (4.20)$$

$$\begin{aligned}
J_3^{\tau\sigma} &= \tau^{\mu\nu} \nabla_\mu R \nabla_\nu \sigma \\
J_4^{\tau\sigma} &= \tau^{\mu\nu} \nabla_\mu \nabla_\nu R \sigma \\
J_5^{\tau\sigma} &= \tau R^2 \sigma \\
J_6^{\tau\sigma} &= \tau \square^2 \sigma \\
J_7^{\tau\sigma} &= \tau R \square \sigma \\
J_8^{\tau\sigma} &= \tau g^{\mu\nu} \nabla_\mu R \nabla_\nu \sigma \\
J_9^{\tau\sigma} &= \tau \square R \sigma
\end{aligned}$$

The coefficients in the result of the δ_τ variation in (4.19) are

$$A_{ij}^{\tau\sigma} = \begin{pmatrix} 0 & 0 & 0 & 0 & -2(x-6) & 0 & 8 & 0 & 0 \\ -12 & -12 & -36 & 6(x-6) & 0 & -2 & -2 & 2 & x-6 \\ 24 & 24 & 72 & -12(x-6) & 2(x-6) & 12 & 4 & 12 & -6(x-6) \end{pmatrix} \quad (4.21)$$

and the coefficients in the δ_σ variations are

$$A_{kj}^{\sigma\tau} = \begin{pmatrix} 2 & 2 & 6 & 6-x & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 6-x & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 4 & 6-x \end{pmatrix} \quad (4.22)$$

Thus, we have obtained a system of 9 equations, which we use to express b_1, b_2, b_3 in terms of c_9, c_{10}, c_{11}

$$\begin{aligned}
b_1 &= 6c_{10} - 12c_{11} \\
b_2 &= 2c_{11} - 2c_9 \\
b_3 &= c_{10} - 6c_{11}
\end{aligned} \quad (4.23)$$

Since Δ_τ does not depend on $B_{\mu\nu\lambda\rho}$, the consistency condition (4.6) is satisfied trivially. In summary, using conditions (4.4) and (4.5), the form of the anomalies is reduced to

$$\begin{aligned}
\Delta_\sigma &= \int d^2x \sqrt{-g} \sum_{j=9}^{11} \sum_{i=1}^{12} c_j M_{ji}^\sigma I_i \\
\Delta_\tau &= \int d^2x \sqrt{-g} \sum_{j=9}^{11} \sum_{k=1}^3 c_j M_{jk}^\tau K_k
\end{aligned} \quad (4.24)$$

where

$$M_{ji}^\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{6-x}{4} & 0 & 1 & 0 & 0 \\ 0 & -2 & -2 & x-6 & 0 & 0 & -6 & 0 & 1 & 0 \\ 0 & 4 & 4 & -2(x-6) & 3 - \frac{x}{2} & \frac{x-6}{4} & 12 & 0 & 0 & 1 \end{pmatrix} \quad (4.25)$$

and

$$M_{jk}^\tau = \begin{pmatrix} 0 & -2 & 0 \\ 6 & 0 & 1 \\ -12 & 2 & -6 \end{pmatrix} \quad (4.26)$$

Now, we check whether the anomalies Δ_σ and Δ_τ are trivial. The most general counterterm C is a linear combination

$$C = \int d^2x \sqrt{-g} \sum_{j=5}^7 d_j C_j \tag{4.27}$$

of the following curvature invariants

$$\begin{aligned} C_5 &= B^{\mu\nu} \nabla_\mu \nabla_\nu R \\ C_6 &= B \square R \\ C_7 &= B R^2 \end{aligned} \tag{4.28}$$

These are the only possible terms if we take into account partial integrations. Variations of δ_σ and δ_τ of C can be expressed as linear combinations of terms I_i and K_k respectively

$$\begin{aligned} \delta_\sigma C &= \int d^2x \sqrt{-g} \sum_{l=5}^7 \sum_{i=1}^{12} d_l A'_{li} I_i \\ \delta_\tau C &= \int d^2x \sqrt{-g} \sum_{l=5}^7 \sum_{k=1}^3 d_l A''_{lk} K_k \end{aligned} \tag{4.29}$$

with coefficients given by

$$A'_{li} = \begin{pmatrix} 0 & -2 & -2 & x-6 & 0 & 0 & -6 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & x-6 & 0 & 0 & -2 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 & x-6 & 0 & -4 & 0 & 0 \end{pmatrix} \tag{4.30}$$

and

$$A''_{lk} = \begin{pmatrix} 6 & 0 & 1 \\ 0 & 0 & 8 \\ 0 & 8 & 0 \end{pmatrix} \tag{4.31}$$

If we take

$$\begin{aligned} d_5 &= c_{10} - 2c_{11} \\ d_6 &= -\frac{c_{11}}{2} \\ d_7 &= \frac{c_{11}}{4} - \frac{c_9}{4} \end{aligned} \tag{4.32}$$

both triviality conditions, (4.7) and (4.8), are satisfied.

Our conclusion is therefore that not only the trace anomalies found in [9] are trivial, but that there cannot be any anomaly whatsoever in $J^{(4)\mu}_{\mu\lambda\rho}$.

5. Conclusion

In this paper we have applied the trace anomaly method to the calculation of moments of Hawking radiation. We have shown that, as suggested in [9] they can be in fact explained

as the fluxes of a W_∞ algebra of chiral currents, which we have constructed out of two chiral scalar field. The non-trivial flux of these currents is generated by their response under a conformal transformation (generalized Schwarzian derivative). Then we have constructed the covariant and Minkowski version of these currents and verified that up to order 6 they are not plagued by any trace anomaly, except for $s = 2$, i.e. for the energy momentum tensor. At this point we have set out to prove that in fact *there cannot exist* any trace anomaly for higher spin currents. We have succeeded in doing so for the fourth order current and we believe this is true also for higher order ones⁴.

The results of this paper are limited to two dimensions. We do not know whether they actually extend to four dimensions. The method of diffeomorphism anomaly to calculate the Hawking radiation, [3], seem to be more general than the trace anomaly method adopted here. It would therefore be very interesting to investigate the use of the latter in order to calculate the higher moments of the Hawking radiation with the same criteria we have used in this paper.

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A. The sixth order current

Here we write down the order 5 and 6 terms corresponding to (3.7):

$$\begin{aligned}
 J_{uuuuu}^{(1,4)} &= -3J_{uu}^{(1,1)}\Gamma^3 - 9J_{uuu}^{(1,2)}\Gamma^2 - 4TJ_{uu}^{(1,1)}\Gamma - 6J_{uuuu}^{(1,3)}\Gamma + \frac{T\hbar(\partial_u T)}{10} + \frac{\hbar(\partial_u^3 T)}{30} \quad (\text{A.1}) \\
 &\quad + j_{uuuuu}^{(1,4)} - (\partial_u T)J_{uu}^{(1,1)} - 4TJ_{uuu}^{(1,2)} \\
 J_{uuuuu}^{(2,3)} &= -\frac{3}{2}J_{uu}^{(1,1)}\Gamma^3 - 3J_{uuu}^{(1,2)}\Gamma^2 - \frac{3}{2}J_{uuu}^{(2,1)}\Gamma^2 - TJ_{uu}^{(1,1)}\Gamma - J_{uuuu}^{(1,3)}\Gamma - 3J_{uuuu}^{(2,2)}\Gamma \\
 &\quad - \frac{T\hbar(\partial_u T)}{30} + \frac{\hbar(\partial_u^3 T)}{60} + j_{uuuuu}^{(2,3)} - TJ_{uuu}^{(2,1)} \\
 J_{uuuuu}^{(3,2)} &= -\frac{3}{2}J_{uu}^{(1,1)}\Gamma^3 - \frac{3}{2}J_{uuu}^{(1,2)}\Gamma^2 - 3J_{uuu}^{(2,1)}\Gamma^2 - TJ_{uu}^{(1,1)}\Gamma - J_{uuuu}^{(3,1)}\Gamma - 3J_{uuuu}^{(2,2)}\Gamma - \frac{T\hbar(\partial_u T)}{30} \\
 &\quad + \frac{\hbar(\partial_u^3 T)}{60} + j_{uuuuu}^{(3,2)} - TJ_{uuu}^{(1,2)} \\
 J_{uuuuu}^{(4,1)} &= -3J_{uu}^{(1,1)}\Gamma^3 - 9J_{uuu}^{(2,1)}\Gamma^2 - 4TJ_{uu}^{(1,1)}\Gamma - 6J_{uuuu}^{(3,1)}\Gamma + \frac{T\hbar(\partial_u T)}{10} + \frac{\hbar(\partial_u^3 T)}{30} \\
 &\quad + j_{uuuuu}^{(4,1)} - (\partial_u T)J_{uu}^{(1,1)} - 4TJ_{uuu}^{(2,1)}
 \end{aligned}$$

⁴So it is not very appropriate to use the term "trace anomaly method". We should rather use the term "Schwarzian derivative method"

and

$$\begin{aligned}
 J_{uuuuuu}^{(1,5)} &= j_{uuuuuu}^{(1,5)} + \hbar \left(\frac{2T^3}{63} + \frac{5}{42} (\partial_u^2 T) T + \frac{17}{168} (\partial_u T)^2 + \frac{1}{42} (\partial_u^4 T) \right) \\
 &\quad - \frac{15}{2} J_{uu}^{(1,1)} \Gamma^4 - 30 J_{uuu}^{(1,2)} \Gamma^3 - 15 T J_{uu}^{(1,1)} \Gamma^2 \\
 &\quad - 30 J_{uuuu}^{(1,3)} \Gamma^2 - 5 (\partial_u T) J_{uu}^{(1,1)} \Gamma - 30 T J_{uuu}^{(1,2)} \Gamma \\
 &\quad - 4 T^2 J_{uu}^{(1,1)} - (\partial_u^2 T) J_{uu}^{(1,1)} - 5 (\partial_u T) J_{uuu}^{(1,2)} - 10 T J_{uuuu}^{(1,3)} - 10 J_{uuuuu}^{(1,4)} \Gamma \\
 J_{uuuuuu}^{(2,4)} &= j_{uuuuuu}^{(2,4)} + \hbar \left(-\frac{2T^3}{63} - \frac{2}{105} (\partial_u^2 T) T - \frac{1}{840} (\partial_u T)^2 + \frac{1}{105} (\partial_u^4 T) \right) \\
 &\quad - 3 J_{uu}^{(1,1)} \Gamma^4 - 9 J_{uuu}^{(1,2)} \Gamma^3 - 3 J_{uuu}^{(2,1)} \Gamma^3 - 4 T J_{uu}^{(1,1)} \Gamma^2 \\
 &\quad - 6 J_{uuuu}^{(1,3)} \Gamma^2 - 9 J_{uuuu}^{(2,2)} \Gamma^2 - (\partial_u T) J_{uu}^{(1,1)} \Gamma \\
 &\quad - 4 T J_{uuu}^{(2,1)} \Gamma - J_{uuuuu}^{(1,4)} \Gamma - 6 J_{uuuuu}^{(2,3)} \Gamma - (\partial_u T) J_{uuu}^{(2,1)} - 4 T J_{uuuu}^{(2,2)} - 4 T J_{uuu}^{(1,2)} \Gamma \\
 J_{uuuuuu}^{(3,3)} &= j_{uuuuuu}^{(3,3)} + \hbar \left(\frac{2T^3}{63} - \frac{1}{70} (\partial_u^2 T) T - \frac{9}{280} (\partial_u T)^2 + \frac{1}{140} (\partial_u^4 T) \right) \\
 &\quad - \frac{9}{4} J_{uu}^{(1,1)} \Gamma^4 - \frac{9}{2} J_{uuu}^{(1,2)} \Gamma^3 - \frac{9}{2} J_{uuu}^{(2,1)} \Gamma^3 - 3 T J_{uu}^{(1,1)} \Gamma^2 \\
 &\quad - \frac{3}{2} J_{uuuu}^{(1,3)} \Gamma^2 - \frac{3}{2} J_{uuuu}^{(3,1)} \Gamma^2 - 9 J_{uuuu}^{(2,2)} \Gamma^2 \\
 &\quad - 3 T J_{uuu}^{(2,1)} \Gamma - 3 J_{uuuuu}^{(2,3)} \Gamma - 3 J_{uuuuu}^{(3,2)} \Gamma \\
 &\quad - T^2 J_{uu}^{(1,1)} - T J_{uuuu}^{(1,3)} - T J_{uuuu}^{(3,1)} - 3 T J_{uuu}^{(1,2)} \Gamma \\
 J_{uuuuuu}^{(4,2)} &= j_{uuuuuu}^{(4,2)} + \hbar \left(-\frac{2T^3}{63} - \frac{2}{105} (\partial_u^2 T) T - \frac{1}{840} (\partial_u T)^2 + \frac{1}{105} (\partial_u^4 T) \right) \\
 &\quad - 3 J_{uu}^{(1,1)} \Gamma^4 - 3 J_{uuu}^{(1,2)} \Gamma^3 - 9 J_{uuu}^{(2,1)} \Gamma^3 - 4 T J_{uu}^{(1,1)} \Gamma^2 \\
 &\quad - 6 J_{uuuu}^{(3,1)} \Gamma^2 - 9 J_{uuuu}^{(2,2)} \Gamma^2 - (\partial_u T) J_{uu}^{(1,1)} \Gamma \\
 &\quad - 4 T J_{uuu}^{(2,1)} \Gamma - J_{uuuuu}^{(4,1)} \Gamma - 6 J_{uuuuu}^{(3,2)} \Gamma - (\partial_u T) J_{uuu}^{(1,2)} - 4 T J_{uuuu}^{(2,2)} - 4 T J_{uuu}^{(1,2)} \Gamma \\
 J_{uuuuuu}^{(5,1)} &= j_{uuuuuu}^{(5,1)} + \hbar \left(\frac{2T^3}{63} + \frac{5}{42} (\partial_u^2 T) T + \frac{17}{168} (\partial_u T)^2 + \frac{1}{42} (\partial_u^4 T) \right) \\
 &\quad - \frac{15}{2} J_{uu}^{(1,1)} \Gamma^4 - 30 J_{uuu}^{(2,1)} \Gamma^3 - 15 T J_{uu}^{(1,1)} \Gamma^2 \\
 &\quad - 30 J_{uuuu}^{(3,1)} \Gamma^2 - 5 (\partial_u T) J_{uu}^{(1,1)} \Gamma - 30 T J_{uuu}^{(2,1)} \Gamma \\
 &\quad - 4 T^2 J_{uu}^{(1,1)} - (\partial_u^2 T) J_{uu}^{(1,1)} - 5 (\partial_u T) J_{uuu}^{(2,1)} - 10 T J_{uuuu}^{(3,1)} - 10 J_{uuuuu}^{(4,1)} \Gamma
 \end{aligned}
 \tag{A.2}$$

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